General Mathematics-related Notes

Mathematics: the science of numbers and their operations, interrelations, combinations, generalizations, and abstractions and of space configurations and their structure, measurement, transformations, and generalizations.

Algebra: a generalization of arithmetic in which letters representing numbers are combined according to the rules of arithmetic.

Calculus: the mathematical methods comprising differential and integral calculus.

Real Numbers: rational numbers + irrational numbers.

Function: a relationship between two sets of elements that associates each element in one set with exactly one element in the other set. In scalar functions, the elements are real numbers. In vector functions, the elements are vectors that may contain real numbers.

Calculus of Variations: a branch of mathematics concerned with applying the methods of calculus to finding the maxima and minima of a function which depends for its values on another function or a curve.

Differential Calculus: a branch of mathematics concerned chiefly with the study of the rate of change of functions with respect to their variables especially through the use of derivatives and differentials.

Integral Calculus: a branch of mathematics concerned with the theory and applications of integrals and integration (as in the determination of lengths, areas, and volumes and in the solution of differential equations).

Deterministic Calculus:

Stochastic Calculus:

Multivariable Calculus:

Set: a collection of elements that have common properties. E.g. {1,2,3,4,5} is a five element set of positive integers.

Space: a set with additional structure. Denoted by a bold, italic, capital letter. E.g. $\boldsymbol{X}$. The points of the space are the elements of the set.

Vector (a.k.a. Linear) Space: a set (of vectors) on which two operations are defined (vector addition and scalar multiplication). These operations satisfy certain natural axioms:

1. Vector addition is associative: $\forall$ u, v, w $\in V$, we have $u + (v + w) = (u + v) + w$.
2. Vector addition is commutative: $\forall$ v, w $\in V$, we have $v + w = w + v$.
3. Vector addition has an identity element: There exists an element $0 \in V$, called the zero vector, such that $v + 0 = v \ \forall \ v \in V$.
4. Vector addition has inverse elements: There exists an element $w \in V$, called the additive inverse of v, such that $v + w = 0 \ \forall \ v \in V$.
5. Distributivity holds for scalar multiplication over vector addition: $a \bullet (v + w) = a \bullet v + a \bullet w \ \forall \ a \in \mathbb{R}$ and $v, w \in V$.
6. Distributivity holds for scalar multiplication over field addition: $(a + b) \bullet v = a \bullet v + b \bullet v \ \forall \ a, b \in \mathbb{R}$ and $v \in V$.
7. Scalar multiplication is compatible with multiplication in the field of scalars: $a \bullet (b \bullet v) = (a \bullet b) \bullet v \ \forall \ a, b \in \mathbb{R}$ and $v \in V$.
8. Scalar multiplication has an identity element: $1 \bullet v = v \ \forall \ v \in V$, where 1 denotes the multiplicative identity in $\mathbb{R}$.

Functional: an association (function) from a vector space to the field underlying the vector space, which is usually the real numbers.

General Mathematics-related Notes

Inner Product Space: a vector space in which distances and angles can be measured = a vector space with the additional structure of an inner product. This additional structure associates each pair of vectors in the space with a scalar quantity known as the inner product of the vectors. Inner products allow the rigorous introduction of intuitive geometrical notions such as the length of a vector or the angle between two vectors. It also provides the means of defining orthogonality between vectors (zero scalar product). Inner product spaces generalize Euclidean spaces (in which the inner product is the dot product, also known as the scalar product) to vector spaces of any (possibly infinite) dimension, and are studied in functional analysis.

Hilbert Space: an inner product space which is "complete", meaning that if a sequence of vectors in the space is Cauchy then it converges to some limit in the space. Hilbert Spaces extend the methods of vector calculus from two and three dimensional spaces to infinite-dimensional spaces.

Cauchy sequence: a sequence (ordered list) whose elements become *arbitrarily close to each other* as the sequence progresses.

Metric Space: a set where a notion of distance (called a metric) between elements of the set is defined. Also, an ordered pair (X,d) where X is a set and d is a metric on X.

Metric: a function $d : X \times X \longrightarrow \mathbb{R}$ such that $\forall$ x,y and z in X

1. $d(x, y) \geq 0$ (*non-negativity*)
2. $d(x, y) = 0$ iff $x = y$ (*identity of indiscernibles*)
3. $d(x, y) = d(y, x)$ (*symmetry*)
4. $d(x, z) \leq d(x, y) + d(y, z)$

Equivalent Metrics: Two metrics d_1 and d_2 on a space X are equivalent if there exist constants $0 < c_1 < c_2 \leq \infty$ such that

$$c_1 d_1(x,y) \leq d_2(x,y) \leq c_2 d_1(x,y) \ \forall (x,y) \in X \times X.$$

Equivalent Metric Spaces: Two metric spaces (X_1,d_1) and (X_2,d_2) are equivalent if there is a function $h : X_1 \longrightarrow X_2$ which is one-to-one and onto (i.e. is invertible) such that the metric $\tilde{d}_1$ on X_1 defined by

$$\tilde{d}_1(x,y) = d_2(h(x),h(y)) \ \forall \ x,y \in X_1$$

is equivalent to d_1.

Fractal Geometry: concerned with the description, classification, analysis and observation of subsets of metric spaces (X,d). Common spaces are the complex plane $\mathbb{C}$, the Rheimann sphere $\hat{\mathbb{C}}$, the code space Σ, a disk in the plane with center at the origin and finite radius ●, an interval in $\mathbb{R}$, the body space in $\mathbb{R}^3$ and the Sierpinski space ⁂.

Geodesics: the concept of shortest paths between points in a space (dependent on the metric).

Real Analysis (theory of functions of a real variable): the branch of mathematics dealing with analytic properties of real functions and sequences, including convergence and limits of sequences of real numbers, the calculus of the real numbers, and continuity, smoothness and related properties of real-valued functions.

Vector Calculus: a field of mathematics concerned with multivariable real analysis of vectors in an inner product space of two or more dimensions; concerned with scalar fields, which associate a scalar to every point in space, and vector fields, which associate a vector to every point in space.

Affine Transformation: a mapping between vector spaces such that vector scaling and linearity is preserved.

One-to-one Mapping: a function $f : X \rightarrow X$ is one-to-one if x,y $\in X$ with $f(x) = f(y)$ implies x = y.

General Mathematics-related Notes

Onto Mapping: a function $f : X \longrightarrow X$ is onto if $f(X) = X$.

Invertible Mapping: a function $f : X \longrightarrow X$ is invertible if it is one-to-one and onto. For an invertible mapping it is possible to define a mapping $f^{-1} : X \longrightarrow X$, called the inverse of f, by $f^{-1}(y) = x$ where $x \in X$ is the unique point such that $y = f(x)$.

Continuous Mapping: A function $f : X_1 \longrightarrow X_2$ from a metric space (X_1,d_1) into a metric space (X_2,d_2) is continuous if, for each $\epsilon > 0$ and $x \in X_1$, there is a $\delta > 0$ so that

$$d_1(x,y) < \delta \Rightarrow d_2(f(x),f(y)) < \epsilon$$

If f is also invertible, and if the inverse f^{-1} is also continuous, then f is a *homeomorphism* between X_1 and X_2 – X_1 and X_2 are *homeomorphic* spaces.

Homeomorphic vs. Equivalent Spaces: The assertion that two spaces are equivalent is much stronger than the statement that they are homeomorphic: to be equivalent there must be a bounded relationship between ϵ and δ independent of x.

Vector Operations:

Gradient: Measures the rate and direction of change in a scalar field. Maps scalar fields to vector fields. E.g. grad(f) = ∇f = vector direction of scalar field, where $\nabla = (\frac{\partial}{\partial \boldsymbol{x}_i},\ldots,\frac{\partial}{\partial \boldsymbol{x}_n})$ is the differential operator "del".

Divergence: Measures the magnitude of a source or sink at a given point in a vector field. Maps vector fields to scalar fields. E.g. $\nabla \bullet \overline{\boldsymbol{F}}$ = scalar source or sink strength

Curl: Measures the tendency to rotate about a point in a vector field. Maps vector fields to vector fields. E.g. $\nabla \times \overline{\boldsymbol{F}}$ = vector field rotational tendency

Laplace operator (Laplacian): A composition of the divergence and gradient operations. Maps scalar fields to scalar fields. $\Delta = \nabla^2 = \nabla \bullet \nabla = \sum_{i=1}^{n} \frac{\partial^2}{\partial \boldsymbol{x}_i^2}$

Tensor Calculus: a field of mathematics concerned with multi-variable real analysis of tensors in an inner product space of two or more dimensions. The classical approach defines a tensor to be a collection of multidimensional arrays, such that one array is associated to each possible coordinate system of any fixed vector space. This notion generalizes scalars, vectors, matrices, linear functionals, etc. To represent a vector *x* as a tensor one can simply let the array associated to any basis *B* be the vector of coordinates of *x* with respect to *B*. Perhaps the most important engineering examples are the stress tensor and strain tensor, which are both 2nd rank tensors, and are related in a general linear elastic material by a fourth rank elasticity tensor. Specifically, a 2nd rank tensor quantifying stress in a 3-dimensional/solid object has components which can be conveniently represented as a 3x3 array. The three Cartesian faces of a cube-shaped infinitesimal volume segment of the solid are each subject to some given force. The force's vector components are also three in number (being in three-space). Thus, 3x3, or 9 components are required to describe the stress at this cube-shaped infinitesimal segment (which may now be treated as a point). Within the bounds of this solid is a whole mass of varying stress quantities, each requiring 9 quantities to describe. Thus, the need for a 2nd order tensor is produced. While tensors can be represented by multi-dimensional arrays of components, the point of having a tensor *theory* is to explain further implications of saying that a quantity is a *tensor*, beyond specifying that it requires a number of indexed components. In particular, tensors behave in specific ways under coordinate transformations. The abstract theory of tensors is a branch of linear algebra (multi-linear algebra).

General Mathematics-related Notes

Differential Equation: a mathematical equation for an unknown function of one variable that relates the values of the function itself and of its derivatives of various orders.

Partial Differential Equation (PDE): a mathematical equation for an unknown function of several variables that relates the values of the function itself and of its derivatives of various orders.

Physically Meaningful Partial Differential Equations:

Heat flow and diffusion (parabolic PDE): $c\frac{\partial u}{\partial t} - \nabla \bullet \left[k \nabla u \right] = f$

where c = material specific heat, vector or scalar function (at most fn(position,time))
u = temperature, vector or scalar function (at most fn(position,time))
k = material diffusivity, vector or scalar function > 0 (at most fn(position))
f = internal heat source, vector or scalar function (at most fn(position,time))

Vibrating systems and wave motion (hyperbolic PDE): $\rho\frac{\partial^2 u}{\partial t^2} - \nabla \bullet \left[k \nabla u \right] = f$

where ρ = material mass density, vector or scalar function (at most fn(position))
u = wave displacement, vector or scalar function (at most fn(position, time))
k = wave propagation speed, vector or scalar function < 0 (at most fn(position))
f = internal wave source, vector or scalar function (at most fn(position,time))

Steady state problems, e.g. bending of elastic beams (elliptical PDE): $\nabla \bullet \left[EI \nabla u \right] = f$

where EI = bending stiffness, vector or scalar function (at most fn(position))
u = transverse beam deflection, vector or scalar function (at most fn(position))
f = load per unit length, vector or scalar function (at most fn(position))

Partial Differential Equations, Solution Techniques:

- Closed Form: exact solution, expressed as a mathematical function
  - Separation of Variables
  - Integral Transforms (e.g. Fourier, Laplace)
  - Change of Coordinates
  - Variable Transformation
- Approximate Solution: when a closed form solution is complicated or not available
  - Finite Difference / Finite Element
  - Perturbation
  - Calculus of Variations
  - Eigenfunction Expansion

Functional Analysis: the branch of mathematics, and specifically of analysis, concerned with the study of vector spaces and operators acting upon them.

Group Theory: A group (G, •) is a set G closed under a binary operation • satisfying the following 3 axioms:

1. Associativity: ∀ a, b and c in G, (a • b) • c = a • (b • c).
2. Identity element: There exists an e ∈ G such that ∀ a in G, e • a = a • e = a.
3. Inverse element: ∀ a in G, there is an element b in G such that a • b = b • a = e, where e is an identity element.